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Effects of Strain-Gradient on the Stress-Concentration at a Cylindrical Hole in a Field of Uniaxial Tension

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SUMMARY

The solution of the plane-strain problem of a circular cylindrical hole in a field of uniaxial tension is obtained in the linear theory of elasticity in which the potential energy function depends on both the strain and the gradient of the strain. The stress-concentration factor at the surface of the cylindrical hole and the stress-concentration away from the hole are found and they are compared with the analogous results obtained in couple-stress theory and in classical elasticity.

Introduction

The classical theory of elasticity of an elastic continuum presupposes that the local state of stress at a material point depends on the corresponding local state of deformation. Such a theory does not accommodate the effects of the atomic structure of solids.

Extensions of the conventional theory which intend to have the local state of stress depend on the local state of deformation and on the deformations in a vicinity of the point in question, began with Cauchy [1]**. The work of Cauchy remained unnoticed until, in 1960, interest in such extended theories was revived by the publications of Aero and Kuvshinskii [2], Grioli [3], Rajagopal [4] and Truesdell and Toupin [5]. All these authors took into account only that part of the first gradient of the strain which constitutes the gradient of the rotation i.e., eight of the eighteen components of the first strain-gradient tensor. This theory is the one Toupin [6] later called the "Cosserat theory with constrained rotations" while Mindlin and Tiersten [7] and Koiter [8], called it the "couple stress theory". The augmentation of the classical theory of elasticity through the inclusion, in the strain energy function, of the complete first gradient of the strain was achieved by Toupin [9].

Later, Mindlin [10] extended the classical theory to include the second gradient of the strain while a further extension, to include all gradients of the strain, was accomplished by Green and Rivlin [11] who called their work the theory of "simple force and stress multipoles". In the language of Toupin [6], the first strain-gradient theory is called the theory of materials of "grade 2", the second strain-gradient theory—the theory of materials of "grade 3", etc.; here the "grade" indicates the order of the space gradients operating on the displacement in the particular theory.

In addition to the strain-gradient theories, another type of extension of the conventional theory exists. The Cosserat brothers [12] introduced a theory of mechanics of continuous media in which a micro-element is embedded in each macro-element (material particle) of the continuum such that the *rotation* of each micro-element may be different from the local rotation of the medium. This is equivalent to a theory of continuous media each point of which has the six degrees of freedom of a rigid body. In the classical theory, a material point has only the three degrees of freedom corresponding to its position in space. If the micro-elements are "frozen" in the corresponding local macro-elements i.e., if the micro rotation is constrained to equal the local rotation of the continuum in the usual sense of elasticity and fluid dynamics, the

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** The existence of this paper was brought to the attension of the first author by R. D. Mindlin.

theory which results is the above mentioned couple-stress theory, or the "Cosserat theory with constrained rotations". An extension of the Cosserat theory in which the micro-elements can *deform* independently of the local deformation of the continuum, was accomplished by Mindlin [13], while a more general theory of this type was given by Green and Rivlin [14]. In the language of Eringen [15], the Cosserat theory is referred to as the theory of "micropolar elasticity" and the couple-stress theory is called the theory of "micropolar elasticity under constrained motion". The equations governing the first strain-gradient theory can be obtained from the micro-structure equations as exhibited by Mindlin [13].

In the present paper we consider the first strain-gradient theory. The general non-linear theory was first given by Toupin [9]. Subsequently, Mindlin [13] derived the linear version of the theory in three-forms—distinguished by different groupings of the eighteen additional variables in the strain energy function. The constitutive equations for the three forms contain *five* new material constants in addition to the conventional elastic pair. So far, no experimental data exists to indicate the numerical value of these new strain-gradient constants. Mindlin [13] has also shown that in the case of mechanically homogeneous, isotropic and centrosymmetric elastic solids, the three versions of the theory yield the same displacement-equations of motion and then exhibited their general solution for cases of statical equilibrium. Later, Mindlin and Eshel [16] derived the relations among the stresses and among the traction boundary conditions for the three forms as well as the necessary and sufficient conditions for positive definiteness of the strain energy function and a theorem of uniqueness of solutions.

The linear first strain-gradient theory differs from the conventional theory of elasticity in several important aspects:

(a) Mindlin's displacement-equations of equilibrium contain two material parameters, l_1, and l_2, having dimensions of length. The presence of these two material length parameters assures the analytical possibility of size effects which are not predicted by the classical theory.

(b) When the length parameters mentioned above tend to zero, one recovers the classical field equations, the classical constitutive equations and the classical boundary conditions. In such a transition to classical elasticity, the order of the governing partial differential equations is lowered and the number of the requisite boundary conditions is diminished, i.e., boundary-layer effects emerge.

In the present paper we consider the stress-concentration problem of a circular cylindrical hole in a mechanically homogeneous, isotropic and centrosymmetric infinite elastic solid subjected, at infinity, to a field of uniaxial tension. Several other stress-concentration problems involving a single cavity have been solved in the context of the first strain-gradient theory: Cook and Weitsman [17], Weitsman[18], Hazen and Weitsman [19]. In the first two papers, enough symmetry is available so as to eliminate some of the material constants from the solution. This simplifies matters and the numerical work can be carried out without having to consider fully the question of the admissible ranges of the remaining material constants. In the last paper this question is examined in full but the authors give an awkward representation of their results, i.e. zones of admissible values of stress. In all these papers, the third form of the theory has been used i.e., that version of the theory in which the eighteen additional variables in the strain energy function are the eight components of the gradient of rotation and the ten components of the fully symmetric part of the gradient of the strain. In the present paper a more convenient form is used i.e., that version of the theory in which the additional variables in the strain energy function are the eighteen components of the first gradient of the strain. The stress-concentration is shown to depend on the radius of the cavity, on the conventional Poisson's ratio and on *four* new material parameters. A complete investigation of the admissible ranges of these new parameters is given and the effect of each of them on the solution is examined.

1. Strain-Gradient Theory—Basic Equations

We now recall, Mindlin [13], the fundamental equations governing the linear strain-gradient theory of homogeneous, isotropic and centrosymmetric elastic solids. In this connection, we confine our attention to the equilibrium case, refer to rectangular cartesian coordinates and use Gibbs's notation and the well known indicial notation.

Let $\boldsymbol{u}$ be the displacement vector; then the kinematic variables are given by:*

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}) = u_{(j,i)} = \varepsilon_{ji} = \text{strain tensor}, \tag{1.1}$$

$$\kappa_{ijk} = \tfrac{1}{2}(u_{k,ji} + u_{j,ki}) = u_{(k,ji)} = \kappa_{ikj} = \text{strain-gradient tensor}. \tag{1.2}$$

The strain energy function assumes the form

$$\begin{aligned} W(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = {} & \tfrac{1}{2}\lambda \varepsilon_{ii}\varepsilon_{jj} + \mu \varepsilon_{ij}\varepsilon_{ij} + a_1 \kappa_{iik}\kappa_{kjj} + a_2 \kappa_{ijj}\kappa_{ikk} \\ & + a_3 \kappa_{iik}\kappa_{jjk} + a_4 \kappa_{ijk}\kappa_{ijk} + a_5 \kappa_{ijk}\kappa_{kji} . \end{aligned} \tag{1.3}$$

We define stresses

$$\tau_{ij} \equiv \frac{\partial W}{\partial \varepsilon_{ij}} = \tau_{ji} = \text{Cauchy stress tensor}, \tag{1.4}$$

$$\mu_{ijk} \equiv \frac{\partial W}{\partial \kappa_{ijk}} = \mu_{ikj} = \text{double stress tensor}, \tag{1.5}$$

where μ_{ijk} have dimensions of force per unit length. These definitions lead to the constitutive equations

$$\tau_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} , \tag{1.6}$$

$$\begin{aligned} \mu_{ijk} = {} & \tfrac{1}{2}a_1(\delta_{ij}\kappa_{kpp} + 2\delta_{jk}\kappa_{ppi} + \delta_{ki}\kappa_{jpp}) \\ & + 2a_2\delta_{jk}\kappa_{ipp} + a_3(\delta_{ij}\kappa_{ppk} + \delta_{ik}\kappa_{ppj}) \\ & + 2a_4\kappa_{ijk} + a_5(\kappa_{kij} + \kappa_{jki}) , \end{aligned} \tag{1.7}$$

where δ_{ij} is the Kronecker-delta, λ and μ are the Lamé constants, $a_1 \dots a_5$ are five new material constants with dimensions of force.

The stress equations of equilibrium appear as

$$\tau_{jk,j} - \mu_{ijk,ij} = 0 \text{ in } R , \tag{1.8}$$

and the natural (traction) boundary quantities to be specified on a smooth bounding surface are

$$\begin{aligned} P_{(n)k} & \equiv n_j(\tau_{jk} - \mu_{ijk,i}) - D_j(n_i\mu_{ijk}) + (D_l n_l) n_i n_j \mu_{ijk} \text{ on } S , \\ R_{(n)k} & \equiv n_i n_j \mu_{ijk} \text{ on } S , \end{aligned} \tag{1.9}$$

where R is the region of space occupied by the material body in question, S is the boundary surface of R, $\boldsymbol{n}$ is the unit outward normal to S, $\boldsymbol{P}_{(n)}$ is the surface force per unit area, $\boldsymbol{R}_{(n)}$ is the surface double force per unit area and D_i (or $\overset{s}{\nabla}$) are the components of the surface gradient:

$$\overset{s}{\nabla}(\ ) \to D_i(\ ) \equiv (\ )_{,i} - n_i n_j (\ )_{,j} .$$

Substituting equations (1.1), (1.2) in (1.6), (1.7) and the result in (1.8), one obtains the displacement equations of equilibrium

$$(\lambda + 2\mu)(1 - l_1^2\nabla^2)\nabla\nabla\cdot\boldsymbol{u} - \mu(1 - l_2^2\nabla^2)\nabla\times\nabla\times\boldsymbol{u} = 0 , \tag{1.10}$$

where

$$l_1^2 = 2\,\frac{a_1 + a_2 + a_3 + a_4 + a_5}{\lambda + 2\mu} , \quad l_2^2 = \frac{a_3 + 2a_4 + a_5}{2\mu} , \tag{1.11}$$

∇ is the gradient operator and l_1 and l_2 are the two material parameters with dimensions of length, mentioned in the introduction.

We further recall, that Mindlin [13] has shown that any solution $\boldsymbol{u}$ of (1.10) can be expressed in terms of a vector function $\boldsymbol{B}$ and a scalar function B_0 according to

$$\boldsymbol{u} = \boldsymbol{B} - l_2^2\nabla\nabla\cdot\boldsymbol{B} - \tfrac{1}{2}(\delta - l_1^2\nabla^2)\nabla[\boldsymbol{r}\cdot(1 - l_2^2\nabla^2)\boldsymbol{B} + B_0] , \tag{1.12}$$

* Here, all quantities without a caret are identical with the respective Mindlin [13] quantities with a caret over them.

where $\boldsymbol{B}$ and B_0 are solutions of

$$(1-l_2^2\nabla^2)\nabla^2\boldsymbol{B}=0\,, \quad (1-l_1^2\nabla^2)\nabla^2 B_0=0 \tag{1.13}$$

respectively, $\delta\equiv(\lambda+\mu)/(\lambda+2\mu)$ and $\boldsymbol{r}$ is the position vector.

Finally, necessary and sufficient conditions for positive definiteness of the strain energy density are, Mindlin and Eshel [16],

$$\begin{aligned}&\mu>0,\quad 3\lambda+2\mu>0,\quad -\bar{d}_1<\bar{d}_2<\bar{d}_1\,,\\ &\bar{a}_2>0,\quad 5\bar{a}_1+2\bar{a}_2>0,\quad 5\bar{f}^2<6(\bar{d}_1-\bar{d}_2)(5\bar{a}_1+2\bar{a}_2)\,,\end{aligned} \tag{1.14}$$

where

$$\begin{aligned}&18\bar{d}_1=-2a_1+4a_2+a_3+6a_4-3a_5\,,\\ &18\bar{d}_2=2a_1-4a_2-a_3,\quad 3\bar{a}_1=2(a_1+a_2+a_3)\,,\\ &\bar{a}_2=a_4+a_5,\quad 3\bar{f}=a_1+4a_2-2a_3\,.\end{aligned} \tag{1.15}$$

2. Formulation of the Problems

The material body under consideration occupies a cylindrical region of space R such that $-\infty<z<\infty$ whose open cross section is D: $r_0<r<\infty$ with a boundary curve C: $r=r_0$.

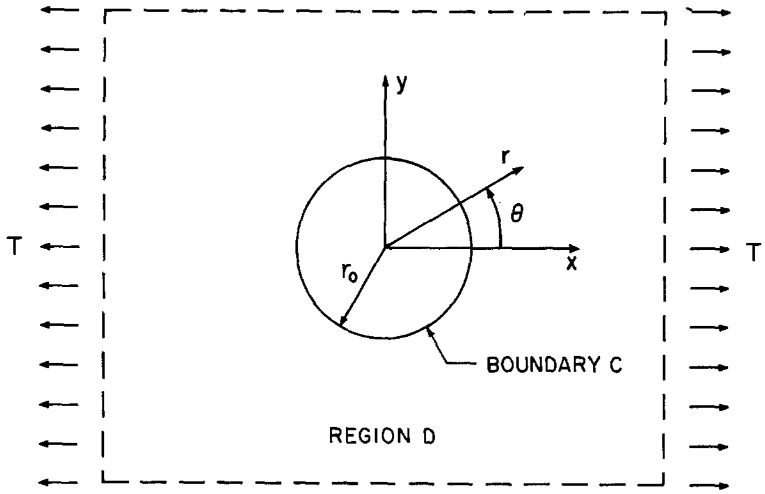

Figure 1. Circular hole in a field of uniaxial tension.

Further, choose cylindrical coordinates r, θ, z such that the z-axis lies along the axis of the cylindrical hole (Fig. 1). Let the body be subjected to a uniaxial tension in the x-direction as shown in Fig. 1, i.e.

$$\begin{aligned}&\text{on } r=r_0: \boldsymbol{\tau}=0,\ \boldsymbol{\mu}=0\,,\\ &\text{as } r\to\infty: \boldsymbol{\tau}\to T\boldsymbol{e}_x\boldsymbol{e}_x=\tfrac{1}{2}T(1+\cos 2\theta)\boldsymbol{e}_r\boldsymbol{e}_r+\tfrac{1}{2}T(1-\cos 2\theta)\boldsymbol{e}_\theta\boldsymbol{e}_\theta-\tfrac{1}{2}T\sin 2\theta(\boldsymbol{e}_r\boldsymbol{e}_\theta+\boldsymbol{e}_\theta\boldsymbol{e}_r)\,,\\ &\boldsymbol{\mu}\to 0\,,\end{aligned} \tag{2.1}$$

where $\boldsymbol{e}_x$, $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ are unit vectors positive in the directions x, r and θ increasing. We now assume that the body is in a state of plane strain parallel to the xy-plane, so that

$$\boldsymbol{u}=u(r,\theta)\boldsymbol{e}_r+v(r,\theta)\boldsymbol{e}_\theta\,,\quad u_z=0\,. \tag{2.2}$$

From this

$$\begin{aligned}&\varepsilon_{rr}=u_{,r},\quad \varepsilon_{r\theta}=\varepsilon_{\theta r}=\tfrac{1}{2}(r^{-1}u_{,\theta}+v_{,r}-r^{-1}v)\,,\\ &\varepsilon_{\theta\theta}=r^{-1}(u+v_{,\theta}),\quad \varepsilon_{\theta z}=\varepsilon_{zr}=\varepsilon_{zz}=0\,,\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\kappa_{rrr} = u_{,rr}, \ \kappa_{r\theta\theta} = -r^{-2}(u+v_{,\theta})+r^{-1}(u_{,r}+v_{,\theta r}),\\
&\kappa_{rr\theta} = \kappa_{r\theta r} = \tfrac{1}{2}[r^{-2}(-u_{,\theta}+v)+r^{-1}(u_{,\theta r}-v_{,r})+v_{,rr}],\\
&\kappa_{\theta rr} = r^{-2}(-u_{,\theta}+v)+r^{-1}(u_{,r\theta}-v_{,r}),\\
&\kappa_{\theta r\theta} = \kappa_{\theta\theta r} = r^{-2}(-u+\tfrac{1}{2}u_{,\theta\theta}-\tfrac{3}{2}v_{,\theta})+r^{-1}(u_{,r}+\tfrac{1}{2}v_{,r\theta}),\\
&\kappa_{\theta\theta\theta} = r^{-2}(2u_{,\theta}+v_{,\theta\theta}-v)+r^{-1}v_{,r},
\end{aligned} \tag{2.4}$$

while all other components are zero.

The non-zero stress components are

$$\begin{aligned}
&\tau_{rr} = (\lambda+2\mu)u_{,r}+\lambda r^{-1}(u+v_{,\theta}), \quad \tau_{\theta\theta} = \lambda u_{,r}+(\lambda+2\mu)r^{-1}(u+v_{,\theta}),\\
&\tau_{r\theta} = \mu r^{-1}(u_{,\theta}-v)+\mu v_{,r}, \quad \tau_{zz} = \lambda r^{-1}(u+v_{,\theta})+\lambda u_{,r},
\end{aligned} \tag{2.5}$$

and from (2.4) and (1.7) one finds the following non-zero double stress components

$$\begin{aligned}
&r_0^{-2}\mu_{rrr} = 2\alpha_1\kappa_{rrr}+\alpha_2\kappa_{r\theta\theta}+\alpha_3\kappa_{\theta\theta r},\\
&r_0^{-2}\mu_{r\theta r} = r_0^{-2}\mu_{rr\theta} = \alpha_4\kappa_{rr\theta}+\tfrac{1}{2}\alpha_5\kappa_{\theta rr}+\tfrac{1}{2}\alpha_3\kappa_{\theta\theta\theta},\\
&r_0^{-2}\mu_{r\theta\theta} = \alpha_2\kappa_{rrr}+2\alpha_6\kappa_{r\theta\theta}+\alpha_5\kappa_{\theta\theta r},\\
&r_0^{-2}\mu_{rzz} = \alpha_2\kappa_{rrr}+2a_2r_0^{-2}\kappa_{r\theta\theta}+a_1r_0^{-2}\kappa_{\theta\theta r},\\
&r_0^{-2}\mu_{\theta rr} = \alpha_5\kappa_{rr\theta}+2\alpha_6\kappa_{\theta rr}+\alpha_2\kappa_{\theta\theta\theta},\\
&r_0^{-2}\mu_{\theta r\theta} = r_0^{-2}\mu_{\theta\theta r} = \tfrac{1}{2}\alpha_3\kappa_{rrr}+\tfrac{1}{2}\alpha_5\kappa_{r\theta\theta}+\alpha_4\kappa_{\theta\theta r},\\
&r_0^{-2}\mu_{\theta\theta\theta} = \alpha_3\kappa_{rr\theta}+\alpha_2\kappa_{\theta rr}+2\alpha_1\kappa_{\theta\theta\theta},\\
&r_0^{-2}\mu_{\theta zz} = a_1r_0^{-2}\kappa_{rr\theta}+2a_2r_0^{-2}\kappa_{\theta rr}+\alpha_2\kappa_{\theta\theta\theta},\\
&r_0^{-2}\mu_{zrz} = r_0^{-2}\mu_{zzr} = \tfrac{1}{2}\alpha_3\kappa_{rrr}+\tfrac{1}{2}a_1r_0^{-2}\kappa_{r\theta\theta}+a_3r_0^{-2}\kappa_{\theta\theta r},\\
&r_0^{-2}\mu_{z\theta z} = r_0^{-2}\mu_{zz\theta} = a_3r_0^{-2}\kappa_{rr\theta}+\tfrac{1}{2}a_1r_0^{-2}\kappa_{\theta rr}+\tfrac{1}{2}\alpha_3\kappa_{\theta\theta\theta},
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
&\alpha_1 = r_0^{-2}(a_1+a_2+a_3+a_4+a_5), \quad \alpha_2 = r_0^{-2}(a_1+2a_2),\\
&\alpha_3 = r_0^{-2}(a_1+2a_3), \quad \alpha_4 = r_0^{-2}(a_3+2a_4+a_5),\\
&\alpha_5 = r_0^{-2}(a_1+2a_5), \quad \alpha_6 = r_0^{-2}(a_2+a_4);
\end{aligned} \tag{2.7}$$

we also define

$$\alpha_0 = r_0^{-2}(a_4+a_5). \tag{2.8}$$

Only four of the seven definitions (2.7), (2.8) are independent; e.g.

$$\alpha_3 = -2\alpha_0+2\alpha_1-\alpha_2, \quad \alpha_5 = 2\alpha_0+2\alpha_1-\alpha_2-2\alpha_4, \quad \alpha_6 = -\alpha_1+\alpha_2+\alpha_4. \tag{2.9}$$

We now note that on $r = r_0$:

$$\boldsymbol{n} = -\boldsymbol{e}_r, \quad \overset{s}{\nabla} = \boldsymbol{e}_\theta\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right). \tag{2.10}$$

Thus, from (1.9), (2.1) and (2.10) the boundary conditions to be specified on $r = r_0$ are:

$$\begin{aligned}
&P_{(r)r} = -\tau_{rr}+\mu_{rrr,r}+r^{-1}(\mu_{\theta rr,\theta}-2\mu_{\theta r\theta}-\mu_{r\theta\theta}) = 0,\\
&P_{(r)\theta} = -\tau_{r\theta}+\mu_{rr\theta,r}+r^{-1}(\mu_{\theta rr}+\mu_{\theta r\theta,\theta}-\mu_{\theta\theta\theta}+\mu_{r\theta\theta,\theta}) = 0,\\
&R_{(r)r} = \mu_{rrr} = 0, \quad R_{(r)\theta} = \mu_{rr\theta} = 0.
\end{aligned} \tag{2.11}$$

As $r \to \infty$, $\boldsymbol{n} = \boldsymbol{e}_r$ which together with equations (1.9) and (2.1) lead to the following boundary conditions:

$$\begin{aligned} &P_{(r)r} = \tau_{rr} \to \tfrac{1}{2}T(1+\cos 2\theta)\,, \quad P_{(r)\theta} = \tau_{r\theta} \to -\tfrac{1}{2}T\sin 2\theta\,, \\ &R_{(r)r} = \mu_{rrr} \to 0\,, \quad R_{(r)\theta} = \mu_{rr\theta} \to 0\,. \end{aligned} \tag{2.12}$$

In cylindrical coordinates, a stress-field of uniaxial tension, T, in the xy-plane is given by

$$\begin{aligned} &\tau^*_{rr} = \tfrac{1}{2}T(1+\cos 2\theta)\,, \quad \tau^*_{r\theta} = -\tfrac{1}{2}T\sin 2\theta\,, \\ &\tau^*_{\theta\theta} = \tfrac{1}{2}T(1-\cos 2\theta)\,, \quad \boldsymbol{\mu}^* = 0\,. \end{aligned} \tag{2.13}$$

In this field, $\boldsymbol{\tau}^*$ and $\boldsymbol{\mu}^*$ assume the correct values as $r\to\infty$, but on $r=r_0$ $\boldsymbol{\tau}^*\neq 0$. We therefore add a stress field $(\boldsymbol{\tau}, \boldsymbol{\mu})$ such that the field $(\boldsymbol{\tau}^*+\boldsymbol{\tau}, \boldsymbol{\mu}^*+\boldsymbol{\mu})$ will satisfy the boundary conditions on $r=r_0$ as well as when $r\to\infty$. It now follows from (2.11), (2.12) and (2.13) that the boundary conditions assume the following form:

on $r=r_0$:

$$\begin{aligned} &-\tfrac{1}{2}T(1+\cos 2\theta) - \tau_{rr} + \mu_{rrr,r} + r^{-1}(\mu_{\theta rr,\theta} - 2\mu_{\theta r\theta} - \mu_{r\theta\theta}) = 0\,, \\ &\tfrac{1}{2}T\sin 2\theta - \tau_{r\theta} + \mu_{rr\theta,r} + r^{-1}(\mu_{\theta rr} + \mu_{\theta r\theta,\theta} - \mu_{\theta\theta\theta} + \mu_{r\theta\theta,\theta}) = 0\,, \\ &\mu_{rrr} = \mu_{rr\theta} = 0\,. \end{aligned} \tag{2.14}$$

As $r\to\infty$:

$$\tau_{rr}\to 0, \quad \tau_{r\theta}\to 0, \quad \mu_{rrr}\to 0, \quad \mu_{rr\theta}\to 0\,. \tag{2.15}$$

For the field $(\boldsymbol{\tau}, \boldsymbol{\mu})$ to be *the* solution, which in turn implies that $(\boldsymbol{\tau}^*+\boldsymbol{\tau}, \boldsymbol{\mu})$ is *the* stress field, it must satisfy equations (1.8), (2.14) and (2.15).

Instead of the stress formulation given above, it is advantageous to consider the displacement formulation of the same problem. Inserting (2.4) in (2.6) and the result together with (2.5) in (2.14), yields boundary conditions in terms of displacement components. Thus, the complete displacement formulation of the boundary value problem in question is:

$$PDE. \quad (\lambda+2\mu)(1-l_1^2\nabla^2)\nabla\nabla\cdot\boldsymbol{\mu} - \mu(1-l_2^2\nabla^2)\nabla\times\nabla\times\boldsymbol{u} = 0 \text{ in } D\,. \tag{2.16}$$

B.C. on $r=r_0$:

$$\begin{aligned} r_0^{-2}P_{(r)r} = {} & u(6r^{-3}\alpha_1 - \lambda r_0^{-2}r^{-1}) + u_{,r}[-6r^{-2}\alpha_1 - r_0^{-2}(\lambda+2\mu)] \\ & + u_{,rrr}2\alpha_1 + u_{,\theta\theta}(-2\alpha_1+2\alpha_2-\alpha_4)r^{-3} + u_{,\theta\theta r}(\alpha_2+\alpha_4)r^{-2} \\ & + v_{,\theta}(8\alpha_1-\alpha_2+\alpha_4)r^{-3} - v_{,\theta}\lambda r_0^{-2}r^{-1} - v_{,\theta r}(4\alpha_1+\alpha_2+\alpha_4)r^{-2} \\ & + v_{,\theta rr}(2\alpha_1-\alpha_4)r^{-1} + v_{,\theta\theta\theta}\alpha_2 r^{-3} \\ = {} & \tfrac{1}{2}Tr_0^{-2}(1+\cos 2\theta)\,, \end{aligned}$$

$$\begin{aligned} r_0^{-2}P_{(r)\theta} = {} & u_{,\theta}(\alpha_0-5\alpha_1+\tfrac{3}{2}\alpha_2-2\alpha_4)r^{-3} - u_{,\theta}\mu r_0^{-2}r^{-1} + u_{,\theta r}(\alpha_0-\alpha_1+\tfrac{3}{2}\alpha_2+2\alpha_4)r^{-2} \\ & + u_{,\theta rr}(2\alpha_1-\tfrac{1}{2}\alpha_4)r^{-1} + u_{,\theta\theta\theta}(\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2-\tfrac{1}{2}\alpha_4)r^{-3} + 2v(-\alpha_0+\alpha_4)r^{-3} \\ & + v\mu r_0^{-2}r^{-1} + 2v_{,r}(\alpha_0-\alpha_4)r^{-2} - v_{,r}\mu r_0^{-2} - \tfrac{1}{2}v_{,rr}\alpha_4 r^{-1} \\ & + v_{,\theta\theta}(-2\alpha_0-6\alpha_1+2\alpha_2+\tfrac{1}{2}\alpha_4)r^{-3} \\ & + v_{,\theta\theta r}(\alpha_0+\alpha_1+\tfrac{1}{2}\alpha_2+\tfrac{1}{2}\alpha_4)r^{-2} + \tfrac{1}{2}v_{,rrr}\alpha_4 \\ = {} & -\tfrac{1}{2}Tr_0^{-2}\sin 2\theta\,, \end{aligned}$$

$$\begin{aligned} r_0^{-2}\mu_{rrr} = {} & 2u(\alpha_0-\alpha_1)r^{-2} + 2u_{,r}(-\alpha_0+\alpha_1)r^{-1} + 2u_{,rr}\alpha_1 \\ & + u_{,\theta\theta}(-\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2)r^{-2} + v_{,\theta}(3\alpha_0-3\alpha_1+\tfrac{1}{2}\alpha_2)r^{-2} \\ & + v_{,\theta r}(-\alpha_0+\alpha_1+\tfrac{1}{2}\alpha_2)r^{-1} \\ = {} & 0\,, \end{aligned}$$

$$\begin{aligned} r_0^{-2}\mu_{rr\theta} &= u_{,\theta}(-3\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2+\tfrac{1}{2}\alpha_4)r^{-2}+u_{,\theta r}(\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2-\tfrac{1}{2}\alpha_4)r^{-1} \\ &\quad +v(2\alpha_0-\tfrac{1}{2}\alpha_4)r^{-2}+v_{,r}(-2\alpha_0+\tfrac{1}{2}\alpha_4)r^{-1}+\tfrac{1}{2}v_{,rr}\alpha_4 \\ &\quad +v_{,\theta\theta}(-\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2)r^{-2} \\ &=0\,. \end{aligned} \tag{2.17}$$

As $r\to\infty$:

the conditions in terms of stresses are given by (2.15). The same conditions but in terms of displacement are furnished by $(2.5)_1$, $(2.5)_3$, $(2.17)_3$ and $(2.17)_4$, i.e.

$$\begin{aligned} &(\lambda+2\mu)u_{,r}+\lambda r^{-1}(u+v_{,\theta})\to 0\,, \qquad \mu r^{-1}(u_{,\theta}-v)+\mu v_{,r}\to 0\,, \\ &2u(\alpha_0-\alpha_1)r^{-2}+2u_{,r}(-\alpha_0+\alpha_1)r^{-1}+2u_{,rr}\alpha_1 \\ &\quad +u_{,\theta\theta}(-\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2)r^{-2}+v_{,\theta}(3\alpha_0-3\alpha_1+\tfrac{1}{2}\alpha_2)r^{-2} \\ &\quad +v_{,\theta r}(-\alpha_0+\alpha_1+\tfrac{1}{2}\alpha_2)r^{-1}\to 0\,, \\ &u_{,\theta}(-3\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2+\tfrac{1}{2}\alpha_4)r^{-2}+u_{,\theta r}(\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2-\tfrac{1}{2}\alpha_4)r^{-1} \\ &\quad +v(2\alpha_0-\tfrac{1}{2}\alpha_4)r^{-2}+v_{,r}(-2\alpha_0+\tfrac{1}{2}\alpha_4)r^{-1}+\tfrac{1}{2}v_{,rr}\alpha_4 \\ &\quad +v_{,\theta\theta}(-\alpha_0+\alpha_1-\tfrac{1}{2}\alpha_2)r^{-2}\to 0\,. \end{aligned} \tag{2.18}$$

3. Solution

We choose

$$\begin{aligned} \boldsymbol{B} &= B(r,\theta)\boldsymbol{e}_x \\ &= [B_1(r,\theta)+B_2(r,\theta)]\,\boldsymbol{e}_x, \quad B_y=B_z=0\,, \\ B_0 &= B_{03}(r,\theta)+B_{04}(r,\theta)+B_{05}(r,\theta)+B_{06}(r,\theta)\,, \end{aligned} \tag{3.1}$$

such that

$$\begin{aligned} B_1 &= A_1 r_0^2 r^{-1}\cos\theta\,, & B_2 &= A_2 r_0 K_1(\rho_2)\cos\theta\,, \\ B_{03} &= A_3 r_0^2 \log r\,, & B_{04} &= A_4 r_0^4 r^{-2}\cos 2\theta\,, \\ B_{05} &= A_5 r_0^2 K_0(\rho_1)\,, & B_{06} &= A_6 r_0^2 K_2(\rho_1)\cos 2\theta\,, \end{aligned} \tag{3.2}$$

where $\rho_1=r/l_1$, $\rho_2=r/l_2$, $K_i(x)$ $(i=0,1,\dots,n)$ are the modified Bessel functions of the second kind of order i and A_j $(j=1,\dots,6)$ are dimensionless constants. The functions $\boldsymbol{B}$ and B_0 above satisfy equations (1.13). Rewriting (1.12) in terms of (3.1) we obtain

$$\begin{aligned} u &= B\cos\theta - l_2^2(\cos\theta\, B_{,r}-r^{-1}\sin\theta\, B_{,\theta})_{,r}+\tfrac{1}{2}[l_1^2\nabla^2(xB_1)-\delta(xB_1+B_0)+B_{02}]_{,r}\,, \\ v &= -B\sin\theta - l_2^2 r^{-1}(\cos\theta\, B_{,r}-r^{-1}\sin\theta\, B_{,\theta})_{,\theta} \\ &\quad + \tfrac{1}{2}r^{-1}[l_1^2\nabla^2(xB_1)-\delta(xB_1+B_0)+B_{02}]_{,\theta}\,. \end{aligned}$$

Inserting (3.2) in (3.3) yields

$$\begin{aligned} u &= \tfrac{1}{2}A_1 r_0^2 r^{-1}-\tfrac{1}{2}A_3\delta r_0^2 r^{-1}-\tfrac{1}{2}A_5(1-\delta)r_0^2 l_1^{-1}K_1(\rho_1) \\ &\quad +\{A_1 r_0^2[\tfrac{1}{2}r^{-1}+2(l_1^2-l_2^2)r^{-3}]-A_2 r_0 l_2 r^{-1}K_2(\rho_2) \\ &\quad +A_4\delta r_0^4 r^{-3}-A_6(1-\delta)r_0^2[r^{-1}K_2(\rho_1)+\tfrac{1}{2}l_1^{-1}K_1(\rho_1)]\}\cos 2\theta\,, \\ v &= \{A_1 r_0^2[-\tfrac{1}{2}(1-\delta)r^{-1}+2(l_1^2-l_2^2)r^{-3}]-A_2 r_0[l_2 r^{-1}K_2(\rho_2)+\tfrac{1}{2}K_1(\rho_2)] \\ &\quad +A_4\delta r_0^4 r^{-3}-A_6(1-\delta)r_0^2 r^{-1}K_2(\rho_1)\}\sin 2\theta\,. \end{aligned} \tag{3.4}$$

This displacement is single valued and is a solution of (2.16). The boundary conditions as $r\to\infty$ are clearly satisfied by (3.4) with any set of finite constants A_1. .... A_6. To satisfy the boundary conditions on $r=r_0$, we insert equations (3.4) and the required derivatives in (2.17), equate coefficients of like functions of θ and obtain a system of six linear algebraic equations in the unknown constants $A_1 \ldots A_6$:

$$\begin{aligned}
&b_{11}B_1+0+b_{13}B_3+0+b_{15}B_5+0=0\,,\\
&b_{21}B_1+b_{22}B_2+0+b_{24}B_4+0+b_{26}B_6=0\,,\\
&b_{31}B_1+b_{32}B_2+0+0+0+b_{36}B_6=-1\,,\\
&b_{41}B_1+0+b_{43}B_3+0+b_{45}B_5+0=-\tfrac{1}{2}\,,\\
&b_{51}B_1+0+0+b_{54}B_4+0+b_{56}B_6=-\tfrac{3}{2}\,,\\
&b_{61}B_1+b_{62}B_2+0+0+0+b_{66}B_6=1\,,
\end{aligned}\tag{3.5}$$

such that

$$B_i=A_i\frac{\mu}{T}\qquad (i=1,\ldots,6)\,,\tag{3.6}$$

and

$$\begin{aligned}
&b_{11}=\beta_1,\quad b_{13}=-\delta\beta_1,\quad b_{15}=(\delta-1)[\tfrac{1}{2}\beta_1 k_1^2K_2(k_1)+\tfrac{1}{4}k_1^3K_1(k_1)]\,,\\
&b_{21}=1-2\delta+\beta_1\left(4\delta\frac{48}{k_1^2}-\frac{48}{k_2^2}\right)+\beta_2\,,\\
&b_{22}=-6\beta_1K_3(k_2)-(\tfrac{1}{2}+\beta_1-\tfrac{1}{2}\beta_2)k_2K_2(k_2),\quad b_{24}=24\beta_1\delta\,,\\
&b_{26}=(\delta-1)[6\beta_1k_1K_3(k_1)+(1+\beta_1)k_1^2K_2(k_1)+\tfrac{1}{2}k_1^3K_1(k_1)]\,,\\
&b_{31}=\frac{2\delta-2\beta_2(2-\delta)}{k_1^2}-\frac{4(1-\delta)}{k_2^2}-\delta\,,\quad b_{32}=\left(\frac{1-\beta_2}{k_1^2}-\frac{2(1-\delta)}{k_2^2}\right)k_2K_2(k_2)+\delta K_1(k_2),\\
&b_{36}=(1-\delta)(1+\beta_2)K_2(k_1)\,,\quad b_{41}=-1\,,\quad b_{43}=\delta\,,\\
&b_{45}=-\delta k_1K_1(k_1),\quad b_{51}=\frac{-6}{k_1^2}(1+\beta_2)-3\delta,\quad b_{54}=-6\delta\,,\\
&b_{56}=(3+3\beta_2-6\delta)K_2(k_1)\,,\quad b_{61}=\delta,\quad b_{62}=-K_1(k_2),\\
&b_{66}=(1-\delta)k_1K_1(k_1)\,,
\end{aligned}\tag{3.7}$$

where

$$\beta_1=\frac{\alpha_0}{2\alpha_1},\quad \beta_2=\frac{\alpha_2}{2\alpha_1},\quad k_1=\frac{r_0}{l_1},\quad k_2=\frac{r_0}{l_2}.\tag{3.8}$$

Upon solving for $B_1 \ldots B_6$ the solution of the problem is completed, and one can find any field quantity desired; specifically, we wish to compute the hoop stress $\tau_{\theta\theta}$. From $(2.5)_2$, $(2.13)_3$, (3.4) and (3.6) one finds the *total* hoop stress as follows

$$\begin{aligned}
\frac{\tau^*_{\theta\theta}+\tau_{\theta\theta}}{T}(r,\theta)=&\tfrac{1}{2}+B_1\frac{r_0^2}{r^2}-B_3\delta\frac{r_0^2}{r^2}+B_5(\delta-1)\left[\frac{r_0}{r}k_1K_1(\rho_1)-\frac{\lambda}{2\mu}k_1^2K_0(\rho_1)\right]\\
&+\left\{-\tfrac{1}{2}+B_1\left(\frac{12}{\rho_1^2}-\frac{12}{\rho_2^2}\right)\frac{r_0^2}{r^2}-B_2\left[\frac{6}{\rho_2}\frac{r_0}{r}K_2(\rho_2)+2\frac{r_0}{r}K_1(\rho_2)\right]\right.\\
&+6B_4\delta\frac{r_0^4}{r^4}+B_6(\delta-1)\left[6\frac{r_0^2}{r^2}K_2(\rho_1)-\frac{\lambda-\mu}{\mu}\frac{r_0}{r}k_1K_1(\rho_1)\right.\\
&\left.\left.-\frac{\lambda}{2\mu}k_1^2K_0(\rho_1)\right]\right\}\cos 2\theta\,.
\end{aligned}\tag{3.9}$$

Noting that

$$\rho_1 = \frac{r}{r_0}\frac{r_0}{l_1} = \frac{r}{r_0}k_1\,, \quad \rho_2 = \frac{r}{r_0}\frac{r_0}{l_2} = \frac{r}{r_0}k_2\,,$$

the hoop stress at $\theta = \pm\frac{1}{2}\pi$ for any r, is then found to be

$$\begin{aligned}\frac{\tau^*_{\theta\theta}+\tau_{\theta\theta}}{T}(r,\theta=\pm\tfrac{1}{2}\pi) = 1 &+ B_1\left[\frac{r_0^2}{r^2} - \frac{r_0^4}{r^4}\left(\frac{12}{k_1^2}-\frac{12}{k_2^2}\right)\right] \\ &+ B_2\left[\frac{r_0^2}{r^2}\frac{6}{k_2}K_2\left(\frac{r}{r_0}k_2\right)+2\frac{r_0}{r}K_1\left(\frac{r}{r_0}k_2\right)\right] - B_3\delta\frac{r_0^2}{r^2} \\ &- 6B_4\delta\frac{r_0^4}{r^4} + B_5(\delta-1)\left[\frac{r_0}{r}k_1K_1\left(\frac{r}{r_0}k_1\right) - \frac{\lambda}{2\mu}k_1^2K_0\left(\frac{r}{r_0}k_1\right)\right] \\ &+ B_6(1-\delta)\left[6\frac{r_0^2}{r^2}K_2\left(\frac{r}{r_0}k_1\right) - \frac{\lambda-\mu}{\mu}\frac{r_0}{r}k_1K_1\left(\frac{r}{r_0}k_1\right)\right. \\ &\left. - \frac{\lambda}{2\mu}k_1^2K_0\left(\frac{r}{r_0}k_1\right)\right]. \end{aligned} \tag{3.10}$$

Finally, the hoop stress on the surface of the cavity at $\theta = \pm\frac{\pi}{2}$ is given as

$$\begin{aligned}\frac{\tau^*_{\theta\theta}+\tau_{\theta\theta}}{T}(r=r_0,\theta=\pm\tfrac{1}{2}\pi) = 1 &+ B_1\left(1-\frac{12}{k_1^2}+\frac{12}{k_2^2}\right) + B_2\left[\frac{6}{k_2}K_2(k_2)+2K_1(k_2)\right] \\ &- B_3\delta - 6B_4\delta + B_5(\delta-1)\left[k_1K_1(k_1)\right. \\ &\left.+\frac{\lambda}{2\mu}k_1^2K_0(k_1)\right] + B_6(1-\delta)\left[6K_2(k_1)\right. \\ &\left.-\frac{\lambda-\mu}{\mu}k_1K_1(k_1) - \frac{\lambda}{2\mu}k_1^2K_0(k_1)\right]. \end{aligned} \tag{3.11}$$

It should be noted here that all the combinations of Lamé constants appearing in (3.7), (3.10) and (3.11) are determined by Poisson's ratio alone, i.e.

$$\frac{\lambda}{2\mu} = \frac{\nu}{1-2\nu}\,, \quad \frac{\lambda-\mu}{\mu} = -\frac{1-4\nu}{1-2\nu}\,, \quad \delta = \frac{1}{2(1-\nu)}\,, \quad 1-\delta = \frac{\mu}{\lambda+2\mu} = \frac{1-2\nu}{2(1-\nu)}\,. \tag{3.12}$$

Whence, we find that the stress-concentration, (3.10) and the stress-concentration factor (3.11), are a function of the five dimensionless parameters ν, k_1, k_2, β_1, β_2. The last two do not appear explicitly in (3.10) and (3.11) but enter in through the boundary conditions on $r = r_0$.

4. Ranges of Material Parameters

To facilitate numerical computations, the stress-concentration factor has to be expressed in terms of independent and dimensionless parameters for which estimates on magnitude can be found.

We define

$$\gamma \equiv \frac{l_1}{l_2} = \left(\frac{2\alpha_1}{\lambda+2\mu}\frac{2\mu}{\alpha_4}\right)^{\frac{1}{2}} = \frac{k_2}{k_1} \tag{4.1}$$

and choose the following as our five independent parameters,

$$\nu = \frac{\lambda}{2(\lambda+\mu)}\,, \quad \beta_1 = \frac{\alpha_0}{2\alpha_1}\,, \quad \beta_2 = \frac{\alpha_2}{2\alpha_1}\,, \quad \gamma = \frac{k_2}{k_1}\,, \quad k_1 = \frac{r_0}{l_1} = \left(\frac{\lambda+2\mu}{2\alpha_1}\right)^{\frac{1}{2}}. \tag{4.2}$$

Upon entering (4.1) in (3.10) and (3.11), one obtains the stress-concentration and the stress-concentration factor, respectively, in terms of (4.2).

Since the numerical values of the new material parameters in the strain-gradient theory are not known from experiment, ranges of (4.2) are found by resorting to the necessary and sufficient conditions for positive definiteness of the strain energy density function given by (1.14) and (1.15).

From $(1.14)_1$ and $(1.14)_2$ the range of Poisson's ratio is chosen to be

$$0 \leqslant \nu < \tfrac{1}{2}. \tag{4.3}$$

In [13, section 12] it is shown that $l_i^2 > 0$ $(i = 1, 2)$. This together with (1.11), $(1.14)_1$, $(1.14)_2$ and the definitions of α_1, α_4 imply that

$$\alpha_1 > 0, \quad \alpha_4 > 0,$$

while $(1.14)_4$, $(1.15)_4$ and the definition of α_0 yield

$$\alpha_0 > 0.$$

From $(1.14)_5$, $(1.15)_3$, $(1.15)_4$ and the definition of α_0, α_1, we find

$$\frac{\alpha_0}{2\alpha_1} < \frac{5}{4}.$$

This, together with the fact that both α_0 and α_1 are positive, fix the range of β_1 as follows

$$0 < \beta_1 < \tfrac{5}{4}. \tag{4.4}$$

The inequalities $(1.14)_3$ and $(1.14)_6$, together with equations (4.1) and (4.2) can now be used to determine the ranges of γ and β_2. This yields

$$0 < \gamma < \left[\frac{15}{16\beta_1}\,\frac{1-2\nu}{1-\nu}\right]^{\frac{1}{2}},$$

$$\beta_2 < 1 - \tfrac{6}{5}\beta_1 + \left[\frac{128}{75}\beta_1^2 - \frac{32}{15}\beta_1 + \frac{1-2\nu}{1-\nu}\,\frac{2}{\gamma^2}\left(1-\tfrac{4}{5}\beta_1\right)\right]^{\frac{1}{2}}, \tag{4.5}$$

$$\beta_2 > 1 - \tfrac{6}{5}\beta_1 - \left[\frac{128}{75}\beta_1^2 - \frac{32}{15}\beta_1 + \frac{1-2\nu}{1-\nu}\,\frac{2}{\gamma^2}\left(1-\tfrac{4}{5}\beta_1\right)\right]^{\frac{1}{2}}.$$

In order to exhibit the effect of each of the parameters on the stress-concentration, it is convenient to assign a "standard" value to each one of the first four parameters in (4.2) and to observe the behavior of the solution as k_1 varies. In addition, variations of the four parameters from their standard values are considered so as to be able to see their effect on the stress-concentration.

The admissible range of each of the parameters as well as their standard values are determined by above inequalities i.e., each set of values assumed by ν, β_1, γ and β_2 satisfies equations (4.3), (4.4) and (4.5).

Furthermore, for the purpose of numerical computation, we impose one more restriction. Comparing the displacement equations of equilibrium, as given by (2.16), with those of the classical theory of elasticity, it is evident that l_1 and l_2 are important contributors to the differences between the two theories. However, it can be shown that if $k_1 \to \infty$ and $k_2 \to \infty$, i.e. $l_1 \to 0$ and $l_2 \to 0$ such that γ and the rest of the parameters remain finite, our solution reduces to the solution of the same problem in classical elasticity. This, together with the fact that results based on the classical theory of elasticity have been substantiated by experiment, lead us to believe that l_1 and l_2 are indeed small as compared to unity. In view of this we select unity as the smallest value of k_1.

Finally, we point out that an examination of equations (4.1) and (4.2) reveals that increasing k_1 while keeping λ, μ, β_1, γ and β_2 constant implies a decrease in $2\alpha_1$ with respect to $(\lambda + 2\mu)$, which in turn requires a decrease of α_4 with respect to 2μ so as to keep γ constant; simultaneously, a decrease in α_0 and α_2 is required in order to keep β_1 and β_2 constant; i.e. the strain-gradient effects disappear as k_1 increases. This observation is compatible with the results mentioned above when passing to the limit as $k_1 \to \infty$ and $k_2 \to \infty$.

5. Numerical Results

The computations are carried out so as to obtain the stress concentration factor as a function of k_1 when only one among the remaining four parameters is allowed to vary, while the others are kept fixed at their respective standard values. The results of these computations, for $\theta = \pm\frac{1}{2}\pi$, are shown in Figures 2–5.

In Fig. 6, the stress-concentration is plotted *versus* the non-dimensional distance r/r_0, where r measures the distance from the center of the cavity along the rays $\theta = \pm\frac{1}{2}\pi$. Results are shown for various values of k_1 while the remaining parameters are kept fixed at their standard values: $\nu = 0.25$, $\beta_1 = 0.50$, $\beta_2 = 0.50$, $\gamma = 0.50$.

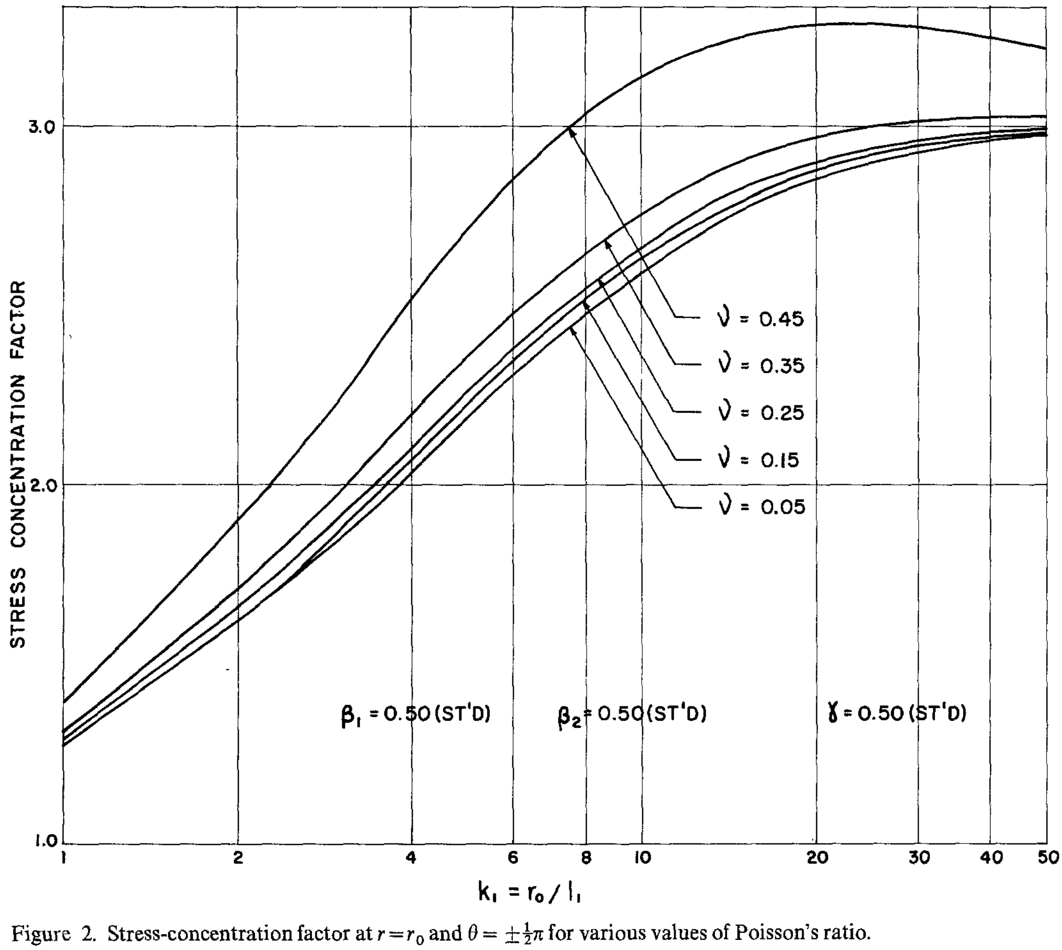

Figure 2. Stress-concentration factor at $r = r_0$ and $\theta = \pm\frac{1}{2}\pi$ for various values of Poisson's ratio.

Examination of Figures 2–5, reveals several interesting characteristics of the solution. Comparing Figure 3 to Figure 4, one finds that the stress-concentration factor at $r = r_0$ and $\theta = \pm\frac{1}{2}\pi$ is only slightly sensitive to changes in β_1 and somewhat more sensitive to variations in β_2 which unlike β_1 can assume negative values. This is manifested when k_1 takes on intermediate values. Comparison of Figures 3 and 4 to Figures 2 and 5 reveals that the stress-concentration factor is much more sensitive to variations in ν and γ than it is to changes in β_1 and β_2. This

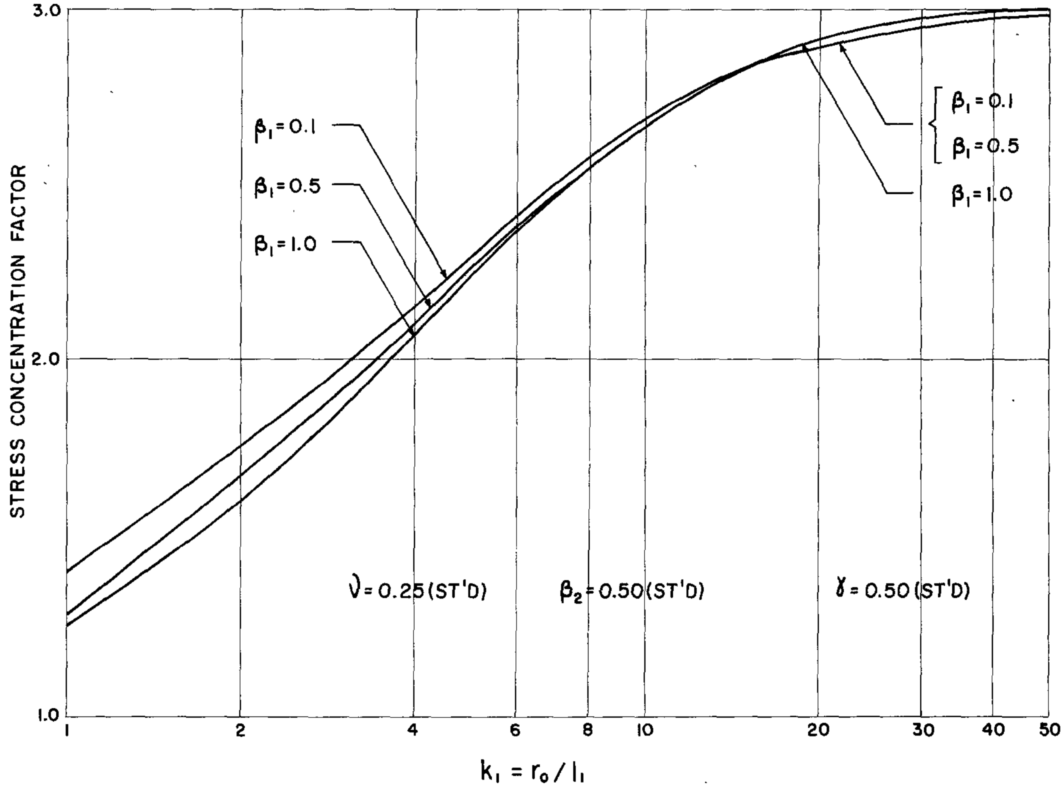

Figure 3. Stress-concentration factor at $r=r_0$ and $\theta=\pm\frac{1}{2}\pi$ for various values of β_1.

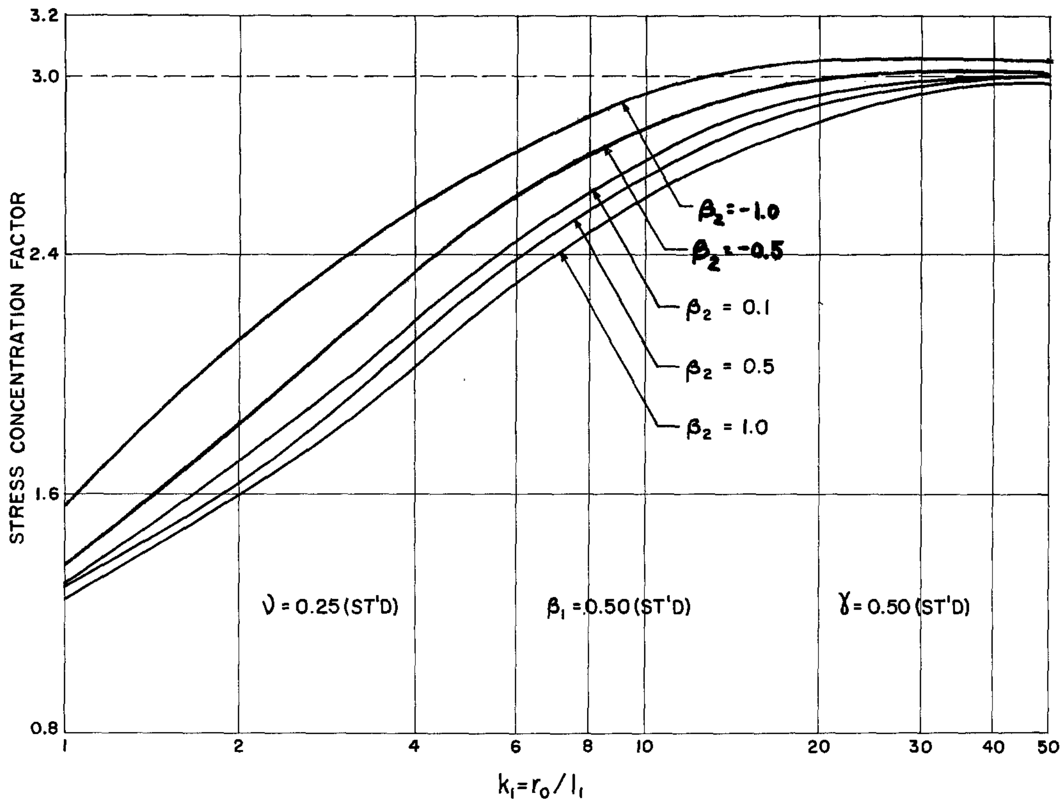

Figure 4. Stress concentration factor at $r=r_0$ and $\theta=\pm\frac{1}{2}\pi$ for various values of β_2.

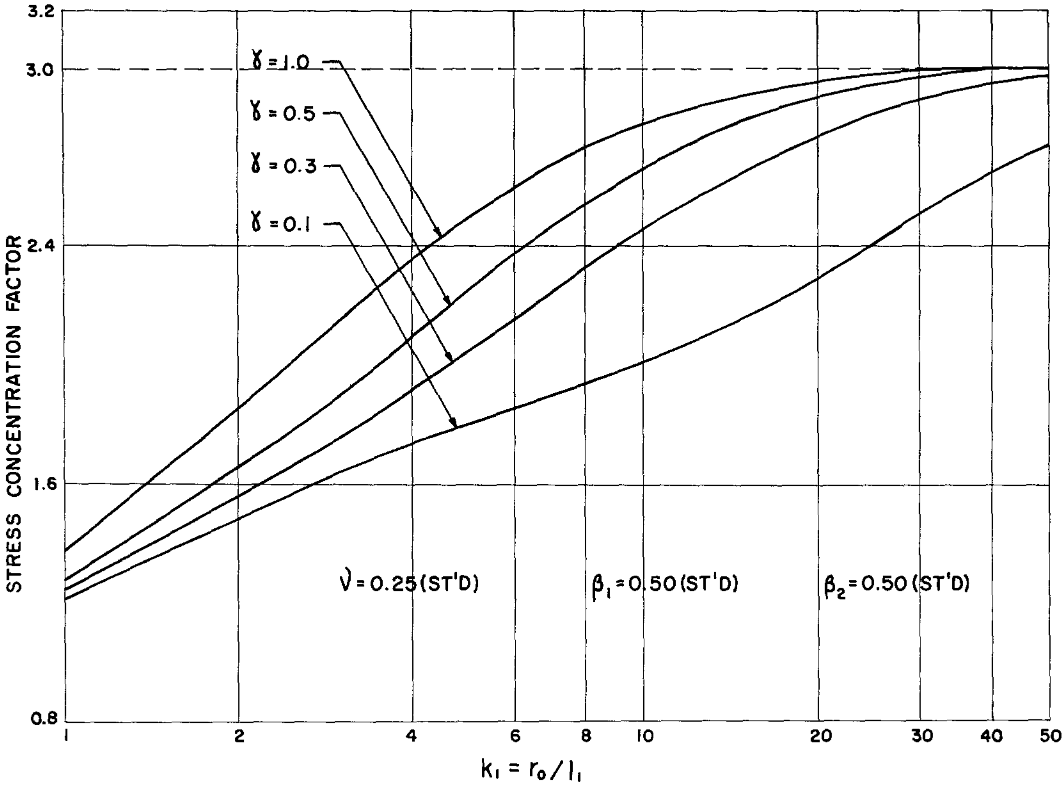

Figure 5. Stress-concentration factor at $r=r_0$ and $\theta=\pm\frac{1}{2}\pi$ for various values of γ.

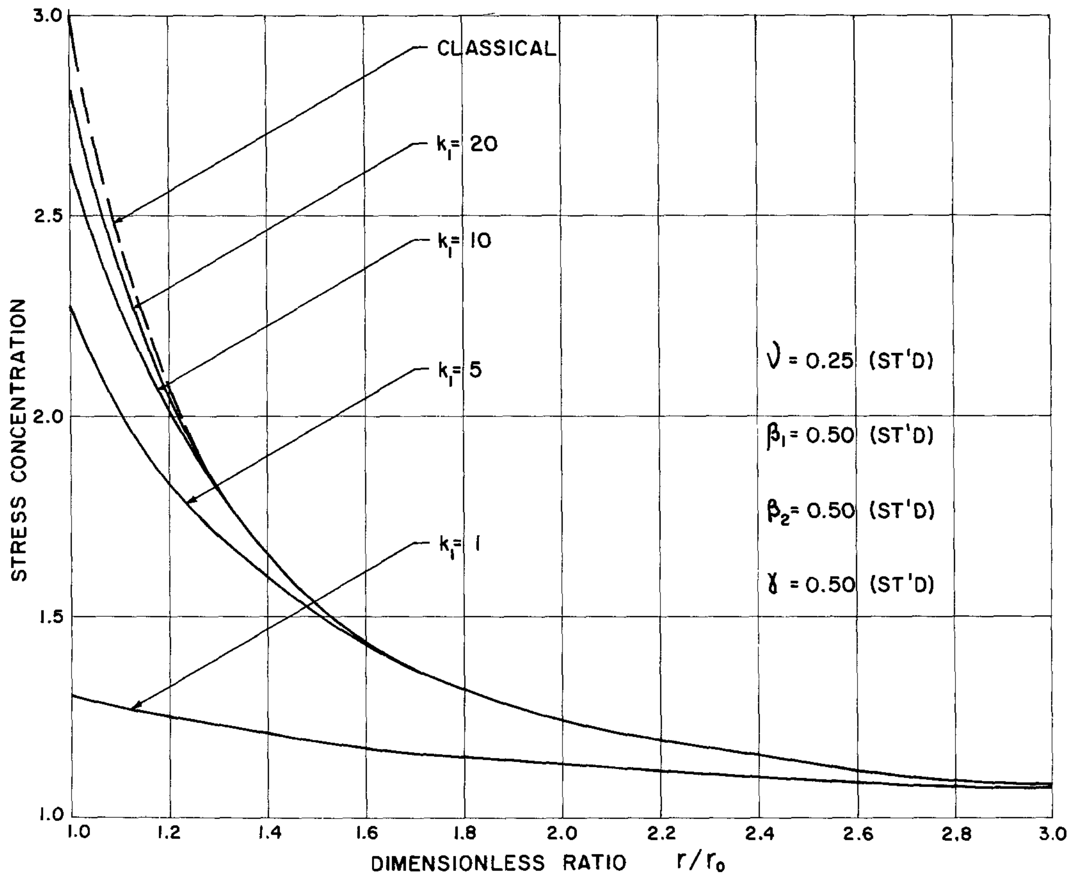

Figure 6. Stress-concentration away from the hole at $\theta=\pm\frac{1}{2}\pi$ for various values of k_1.

is clearly manifested when k_1 takes on intermediate values. It is interesting to note that no ratio of couple-stress constants analogous to β_1 and β_2 appears in the Mindlin and Tiersten's [7] result for the cylindrical hole; but one does appear in their stress-concentration factor for the spherical cavity with a similar insensitivity. All this is clearly in contrast to the solution obtained in classical elasticity, where the stress-concentration factor at $r=r_0$ assumes a maximum value of 3 independent of material properties and the radius of the cavity.

Consideration of Figures 2,3,4, and 5 shows that the stress-concentration factor approaches 3 as k_1 increases. This confirms Mindlin and Tiersten's main result. However, Figures 2 and 4 reveal that when Poisson's ratio is greater than about $\frac{1}{3}$ and when β_2 assumes negative values, the stress-concentration factor may exceed 3; in fact, it may reach a max. value at an intermediate value of k_1, and then approach the classical value of 3 from above as clearly seen when $\nu=0.45$.

Examination of Fig. 6 reveals that the solution of classical elasticity and the present solution become indistinguishable as r/r_0 increases.

Comparing the results of the present paper with those of Hazen and Weitsman [19], we observe that in the case of a cylindrical hole in a field of uniaxial tension, only *four* new material parameters (i.e. four ratios of material constants) appear in the solution although there are *five* new material constants in the constitutive equations. In the case of a spherical cavity in a field of uniaxial tension [19], *five* new material parameters appear.

While the results of this paper confirm the main results given in [19], the representation of results in "zones of admissible values of stress" is avoided here. The choice of the parameters and the numerical computations have been carried out so that the behavior of the hoop stresses as a function of the material parameters could be clearly exhibited.

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